Pergamon
www.elsevier.com/locate/jappmathmech
PII: S0021-8928(02)00043-6
J. Appl. Maths Mechs, Vol. 66, No. 3, pp. 341–345, 2002

0021-8928/02/\$—see front matter

THE STABILITY OF SYSTEMS WITH DISTRIBUTED PARAMETERS AND LUMPED FORCES†

F. D. BAIRAMOV and M. Yu. SAFRONOV

Naberezhnyye Chelny

(*Received* 23 *March* 2001)

The Lyapunov-function method is used to investigate the stability of systems with distributed parameters and lumped forces described by linear partial differential equations (for example, elastic structures with lumped masses, dampers, elastic aircraft with rigid control rudders, etc.). By introducing additional variables, the initial equations of high order are represented by a system of evolution equations and constraint equations, which are first-order partial differential equations. At the points where the lumped forces are applied, certain phase functions experience a discontinuity of the first kind and matching conditions are satisfied. A method for stability investigation is developed for these systems. The change to first-order equations helps to construct the Lyapunov functionals. As an example, the stability of torsional vibrations of an elastic aerofoil with a suspended engine is considered. © 2002 Elsevier Science Ltd. All rights reserved.

1. STATEMENT OF THE PROBLEM

Consider a system of first-order partial differential equations

$$\frac{\partial\varphi}{\partial t}=A\frac{\partial\varphi}{\partial x}+B\frac{\partial\psi}{\partial x}+A_0\varphi+B_0\psi,\quad C\frac{\partial\varphi}{\partial x}+D\frac{\partial\psi}{\partial x}+C_0\varphi+D_0\psi=0 \tag{1.1}$$

$$x\in(0,1)\ \ x_i\neq 0,\ \ i=\overline{1,f},\ \ t\in I=[0,\infty)$$

where $\varphi = \varphi(x, t)$ is an n-dimensional vector of phase functions, $\varphi = \varphi(x, t)$ is an s-dimensional vector of phase functions whose time derivatives do not occur in system (1,1), $A(x)$, $B(x)$, $C(x)$ and $D(x)$ are matrices whose elements are bounded measurable functions together with their first derivatives, and $A_0(x)$, $B_0(x)$,
$C_0(x)$ and $D_0(x)$ are matrices whose elements are bounded measurable functions.

Any linear partial differential equation of arbitrary order, or system of such equations, may be reduced to the form of (1.1) by introducing additional variables [1, 2]. The second equation of (1.1) appears not only because the order of the partial derivatives is reduced, but also owing to those equations without time derivatives that may occur in the initial system. An example is the equation of continuity when describing incompressible fluid flow, etc.

The following homogeneous boundary conditions are given at $x = 0$ and $x = 1$

$$\Gamma_1\varphi(0,t)+\Gamma_2\psi(0,t)=0,\quad \Gamma_3\varphi(1,t)+\Gamma_4\psi(1,t)=0 \tag{1.2}$$

where $\Gamma_1, \dots \Gamma_4$ are matrices of constants.

Suppose at points $x = x_i (i = \overline{1,f})$ the system is subject to lumped forces, which depend on the phase functions and, in the case of forces of inertia, also on the first time derivatives of the phase functions. At these points some of the phase functions experience a discontinuity of the first kind and one has the following matching conditions.

$$\chi'(x,t)=\chi''(x_i,t)+K_{1i}\chi''(x_i,t)+K_{2i}\frac{d\varphi''(x_i,t)}{dt},\quad i=\overline{1,f} \tag{1.3}$$

where

$$\chi(x,t)=\left\|\begin{matrix}\varphi(x,t)\\ \psi(x,t)\end{matrix}\right\|;\quad \varphi'(x_i,t),\quad \varphi''(x_i,t)$$

†*Prikl. Mat. Mekh.* Vol. 66, No. 3, pp. 350–355, 2002.

$\chi'(x_i, t)$, $\varphi''(x_i, t)$ and $\chi'(x_i, t)$, $\chi''(x_i, t)$ are the limits of the functions $\varphi(x, t)$ and $\chi(x, t)$ as $x \to x_i$ from the left and right, respectively, and K_{1i} and $K_2 i(i = \overline{1,f})$ are $(n + s) \times (n + s)$ constant coefficient matrices, which depend on the form of the applied forces.

The solution of system (1.1)–(1.3) in the intervals between the points x_i with given initial data $\varphi_0(x, t)$ is considered in the class of functions which are continuously differentiable with respect to t and continuous and differentiable almost everywhere with respect to x; at the points of discontinuity x_i, Eqs (1.1) are replaced by the matching conditions (1.3) [3]. The trivial solution $\varphi = \psi = 0$ corresponds to the unperturbed state.

We shall consider the stability of the solution $\varphi = \psi = 0$ of system (1.1)–(1.3) with respect to two measures [4]

$$\rho = \int_0^1 \varphi^T \varphi dx, \quad \rho_0 = \int_0^1 \varphi^T(x,t)\varphi(x,t)dx + \sum_{i=1}^{f} \varphi''^T(x_i,t)\varphi''(x_i,t)$$

where ρ_0 and ρ_0 constrain the actual and initial perturbations, respectively.

2. INVESTIGATION OF STABILITY

To solve the problem formulated above, we shall construct a Lyapunov function in the form

$$V = \sum_{i=1}^{f+1} \int_{x_{i-1}}^{x_i} \varphi^T \upsilon_i(x)\varphi dx + \sum_{i=1}^{f} \varphi''^T(x_i,t)\omega_i \varphi''(x_i,t) \tag{2.1}$$

Where $\upsilon_i(x)(i = \overline{1,f+1})$, $\omega_i(i = \overline{1,f})$ are $n \times n$ symmetric matrices; the elements of the matrices $\upsilon_i(x)(i = \overline{1,f+1})$ are continuous functions, differentiable almost everywhere with respect to x for $x \in (x_{i-1}, x_i)$; $x_0 = 0, x_{f+1} = 1$.

Using the technique introduced in [2], we find the derivative of the functional V along trajectories of the first equation of (1.1), which is

$$\begin{aligned} \frac{dV}{dt} = \sum_{i=1}^{f+1} \int_{x_{i-1}}^{x_i} \{\varphi^T \upsilon_i\left(A\frac{\partial \varphi}{\partial x} + B\frac{\partial \psi}{\partial x}\right) + \left(\frac{\partial \varphi^T}{\partial x}A^T + \frac{\partial \psi^T}{\partial x}B^T\right)\upsilon_i \varphi + \varphi^T(\upsilon_i A_0 + A_0^T \upsilon_i)\varphi + \\ + \varphi^T \upsilon_i B_0 \psi + \psi^T B_0^T \upsilon_i \varphi\}dx + \sum_{i=1}^{f} 2\varphi''^T(x_i,t)\omega_i \frac{d\varphi''(x_i,t)}{dt} \end{aligned} \tag{2.2}$$

Following the method of Lagrange multipliers, we add to this derivative the expression

$$\begin{aligned} &\sum_{i=1}^{f+1} \int_{x_{i-1}}^{x_i} (\varphi^T P_{1i} + \psi^T P_{2i})\left\{\left(C\frac{\partial \varphi}{\partial x} + D\frac{\partial \psi}{\partial x}\right) + C_0\varphi + D_0\psi\right\}dx + \\ &+ \sum_{i=1}^{f+1} \int_{x_{i-1}}^{x_i} \left\{\left(\frac{\partial \varphi^T}{\partial x}C^T + \frac{\partial \psi^T}{\partial x}D^T\right) + \varphi^T C_0^T + \psi^T D_0^T\right\}(P_{1i}^T\varphi + P_{2i}^T\psi)dx \end{aligned} \tag{2.3}$$

which vanishes by virtue of the second equation of (1.1). Here $P_{1i} = P_{1i}(x)$ and $P_{2i} = P_{2i}(x)$ $(i = \overline{1,f+1})$ are as yet arbitrary $n \times n'$ and $s \times s'$ matrices whose elements are functions that are continuous and differentiable almost everywhere with respect to x. The expressions $(\varphi^T P_{1i} + \psi^T P_{2i})$ and $(P_{1i}^T\varphi + P_{2i}^T\psi)$ $(i = \overline{1,f+1})$ in parentheses in (2.3) play the role of Lagrange multipliers.

Integrating by parts, we obtain

$$\begin{aligned} \frac{dV}{dt} = \sum_{i=1}^{f+1} \int_{x_{i-1}}^{x_i} &\left\{-\varphi^T w_i \varphi + \psi^T\left(P_{2i}D_0 + D_0^T P_{2i}^T - \frac{\partial P_{2i}D}{\partial x}\right)\psi + \right. \\ &+ \varphi^T\left[P_{1i}D_0 + C_0^T P_{2i}^T + \upsilon_i B_0 - \frac{\partial(\vartheta_i B + P_{1i}D)}{\partial x}\right]\psi + \\ &+ \psi^T\left[D_0^T P_{1i}^T + P_{2i}C_0 + B_0^T \upsilon_i - \frac{\partial(B^T\upsilon_i + D^T P_{1i}^T)}{\partial x}\right]\varphi + \end{aligned}$$

$$+\frac{\partial\varphi^T}{\partial x}(A^T\upsilon_i+C^TP_{1i}^T-\upsilon_iA-P_{1i}C)\varphi+\frac{\partial\psi^T}{\partial x}(D^TP_{2i}^T-P_{2i}D)\psi+$$

$$+\frac{\partial\varphi^T}{\partial x}(C^TP_{2i}^T-\upsilon_iB-P_{1i}D)\psi+\psi^T(P_{2i}C-B^T\upsilon_i-D^TP_{1i}^T)\frac{\partial\varphi}{\partial x}\Bigg\}dx+$$

$$+\sum_{i=1}^{f}2\varphi''^T(x_i,t)\omega_i\frac{d\varphi''^T(x_i,t)}{dt}+$$

$$+\sum_{i=1}^{f+1}[\chi'^T(x_i,t)Q_i(x_i,t)\chi'(x_i,t)-\chi''^T(x_{i-1},t)Q_i(x_{i-1},t)\chi''(x_{i-1},t))] \tag{2.4}$$

$$Q_i(x)=\begin{Vmatrix}\upsilon_i(x)A(x)+P_{1i}(x)C(x) & \upsilon_i(x)B(x)+P_{1i}(x)D(x)\\ B^T(x)\upsilon_i(x)+D^T(x)P_{1i}^T(x) & P_{2i}(x)D(x)\end{Vmatrix},\quad i=\overline{1,f+1}$$

$$w_i=\frac{\partial(\upsilon_iA+P_{1i}C)}{\partial x}-\upsilon_iA_0-A_0^T\upsilon_i-P_{1i}C_0-C_0^TP_{1i}^T,\quad i=\overline{1,f+1}$$

Suppose the matrices υ_i, P_{1i} and P_{2i} satisfy the equations

$$\begin{gathered}\upsilon_iA+P_{1i}C=A^T\upsilon_i+C^TP_{1i}^T,\quad P_{2i}D=D^TP_{2i}^T\\ \upsilon_iB+P_{1i}D=C^TP_{2i}^T,\quad P_{2i}D_0+D_0^TP_{2i}^T=\frac{\partial P_{2i}D}{\partial x}\\ \upsilon_iB_0+P_{1i}D_0+C_0^TP_{2i}^T=\frac{\partial(\upsilon_iB+P_{1i}D)}{\partial x};\quad x\in(x_{i-1},x_i),\ i=\overline{1,f+1}\\ K_{2i}Q_i(x_i)K_{2i}=0,\quad i=\overline{1,f}\end{gathered} \tag{2.5}$$

and boundary conditions

$$\chi^T(x_0,t)Q_1(x_0,t)\chi(x_0,t)=0,\ \chi^T(x_{f+1},t)Q_{f+1}(x_{f+1},t)\chi(x_{f+1},t)=0 \tag{2.6}$$

Substituting $\chi'(x_i,t)$ from Eq. (1.3) and $d\varphi''(x_i,t)/dt$ from the first of equations (1.1) into relation (2.4), we obtain

$$\frac{dV}{dt}=-\sum_{i=1}^{f+1}\int_{x_{i-1}}^{x_i}\varphi^T(x,t)w_i\varphi(x,t)dx-\sum_{i=1}^{f}\chi''^T(x_i,t)h_i\chi''(x_i,t)+$$

$$+2\sum_{i=1}^{f}\chi''^T(x_i,t)\,\|\,R_iA\quad R_iB\,\|\left.\frac{\partial\chi''^T(x_i,t)}{\partial x}\right|_{x\to x_i+0} \tag{2.7}$$

$$R_i=\begin{Vmatrix}\omega_i\\0\end{Vmatrix}+(K_{1i}^T+E)Q_i(x_i)\begin{Vmatrix}K_{2i}\\0\end{Vmatrix}$$

$$h_i=-\|\,R_iA_0\quad R_iB_0\,\|-\begin{Vmatrix}A_0^TR_i^T\\B_0^TR_i^T\end{Vmatrix}-K_{1i}^TQ_i(x_i,t)-K_{1i}^TQ_i(x_i,t)K_{1i}-Q_i(x_i,t)K_{1i}-$$

$$-Q_i(x_i,t)+Q_{i+1}(x_i,t)$$

(E is the identity matrix).

In order that the derivative dV/dt (2.7) should not contain terms with a derivative with respect to x, we choose matrices ω_i ($i=\overline{1,f}$) subject to the condition

$$R_iA(x_i)=R_iB(x_i)=0,\quad i=\overline{1,f} \tag{2.8}$$

By the Lyapunov function method [5], the trivial solution of system (1.1)–(1.3) will be asymptotically stable with respect to the two measures ρ and ρ_0 if the functional V (2.1) is continuous with respect to ρ_0 and positive-definite with respect to ρ, and its derivative dV/dt is negative-definite with respect to ρ.

The continuity of the functional V (2.1) with respect to ρ_0 follows directly from the continuity and therefore boundedness of the elements of the matrix $\upsilon_i(x)$ for $x \in (x_{i-1}, x_i)$ $(i = \overline{1, f+1})$. The remaining conditions of this assertion will be satisfied if the matrices ω_i and h_i are non-negative, and the matrices $\upsilon_i(x)$ and $w_i(x)$ are positive-definite for any $x \in (x_{i-1}, x_i)$, that is

$$\omega_i \geqslant 0,\ h_i \geqslant 0,\quad i = \overline{1,f};\quad \upsilon_i(x) > 0,\ w_i(x) > 0,\quad x \in (x_{i-1}, x_i),\quad i = \overline{1, f+1} \tag{2.9}$$

Example. Let us consider the stability of torsional vibrations of an aerofoil with a suspended engine, as described by the following equation and boundary conditions in dimensionless form

$$J\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial x}\left(GI\frac{\partial y}{\partial x}\right) - M_1 y - M_2\frac{\partial y}{\partial x},\quad x \in (0,1),\quad x \neq x_* \tag{2.10}$$

$$y(0,t) = \left.\frac{\partial y(x,t)}{\partial t}\right|_{x=1} = 0,\quad \lim_{x\to x_*-0} y(x,t) = \lim_{x\to x_*-0} y(x,t) - J_*\frac{d^2 y(x_*,t)}{dt^2}$$

where

$$t = t_0\sqrt{\frac{g}{l}},\quad x = \frac{z}{l},\quad x_* = x_1 = \frac{z_*}{l},\quad GI = \frac{G_0 I_0}{m_k g l^2},\quad J = \frac{J_0}{m_k l^2}$$

$$M_1 = \frac{M_{10}}{m_k g},\quad M_2 = \frac{M_{20}}{m_k g},\quad J_* = \frac{J_{*0}}{m_k l^2}$$

t_0 is the time, l is the length of the aerofoil panel, z are the coordinates of the aerofoil points, z_* is the coordinate of the place at which the engine is attached, $y = y(x, t)$ is the torsion angle of the aerofoil section with coordinate x, g is the acceleration due to gravity, $G_0 I_0 = G_0 I_0(x)$ is the torsional stiffness of the aerofoil, m_k is the mass of the aerofoil $M_{10} = M_{10}(x)$ is the coefficient of the linear moment of aerodynamic forces, $M_{20} = M_{20}(x)$ is the coefficient of aerodynamic damping, J_{*0} is the moment of inertia of the engine relative to the stiffness axis of the aerofoil and $J_0 - J_0(x)$ is the distributed moment of inertia of the aerofoil about the stiffness axis.

Introducing new variables, we write Eqs (2.10) in the form of system (1.1)–(1.3), where

$$\varphi_1 = y(x,t),\quad \varphi_2 = \frac{\partial y(x,t)}{\partial t},\quad \varphi_3 = GI\frac{\partial y(x,t)}{\partial x}$$

$$A = \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & 1/J \\ 0 & GI & 0 \end{Vmatrix},\quad A_0 = \begin{Vmatrix} 0 & 1 & 0 \\ -M_1/J & -M_2/J & 0 \\ 0 & 0 & 0 \end{Vmatrix},\quad K_{21} = \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -J_* & 0 \end{Vmatrix}$$

$$C = \begin{Vmatrix} 1 & 0 & 0 \end{Vmatrix},\quad C_0 = \begin{Vmatrix} 0 & 0 & \dfrac{1}{GI} \end{Vmatrix},\quad \Gamma_1 = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{Vmatrix},\quad \Gamma_3 = \begin{Vmatrix} 0 & 0 & 1 \end{Vmatrix}$$

and B, B_0, D, D_0, Γ_2, Γ_4 and K_{11} are zero matrices. Using Eqs (2.4)–(2.8), we construct matrices $\upsilon_i(x)$, $f_i(x)(i = 1, 2)$, ω and h with the following elements

$$\upsilon_i^{(11)} = M_2 + M_1 c_2,\quad \upsilon_i^{(21)} = \upsilon_i^{(12)} = J(x),\quad \upsilon_i^{(31)} = \upsilon_i^{(13)} = (i-2)J_*(GIc_2)$$

$$\upsilon_i^{(22)} = J(x)c_2,\quad \upsilon_i^{(32)} = \upsilon_i^{(23)} = 2(i-1)(1-x)\tilde{J}_* J(x)/(GI),\quad \upsilon_i^{(33)} = c_2/(GI)$$

$$f_i^{(11)} = 2M_1,\quad f_i^{(22)} = 2M_2 c_2 - 2J(x) + (i-1)\tilde{J}_* d((1-x)J(x))/dx$$

$$f_i^{(13)} = 2(i-1)(1-x)\tilde{J}_* M_1,\quad f_i^{(23)} = 2(i-1)(1-x)\tilde{J}_* M_2/(GI)$$

$$f_i^{(33)} = 2/(GI) + 2(i-1)\tilde{J}_* d((1-x)/(GI))/dx,\quad x \in (x_{i-1}, x_i),\quad i = 1,2$$

$$\tilde{J}_* = J_*/((1-x_1)J(x_1));\quad \omega^{(11)} = J_*/c_2,\quad \omega^{(12)} = J_*,\quad \omega^{(22)} = J_* c_2$$

$$h^{(33)} = 2J_*/(GI(x_1)J(x_1))$$

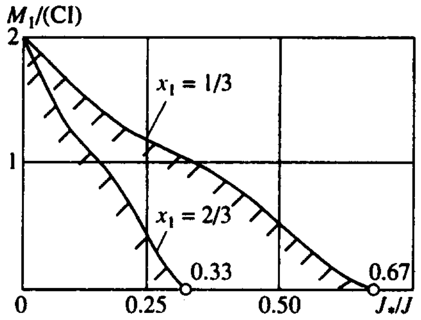

Fig. 1

where $f_i^{(kj)}$, $\upsilon_i^{(kj)}$, $\omega^{(kj)}$ and $h^{(kj)}$ are the elements of the matrices f_i, υ_i, ω and h, respectively. The remaining elements of f_i, υ_i ω and h are zero.

Using the inequality [2]

$$2\int_0^1 \varphi_1^2 dx \leqslant \int_0^1 \left(\frac{\varphi_3}{GI}\right)^2 dx \leqslant \frac{1}{\min\limits_x GI}\int_0^1 \frac{\varphi_3^2}{GI} dx$$

and taking into account that the constant c_2 was chosen arbitrarily, we obtain from relations (2.9) following conditions for asymptotic stability of torsional vibrations of the aerofoil

$$\min_x M_2 > 0, \quad 1 + \tilde{J}_* \xi > 0$$

$$\min_x M_1 + 4 \min_x (GI)(1 + \tilde{J}_* \xi)/(1 + \sqrt{1 + 16 J_* / J(x_1)}) > 0$$

$$\xi = \min_{x_1 < x < 1} [GI d((1-x)/(GI))/dx] > 0$$

For an aerofoil of constant profile (GI, J, M_1, M_2 = const), the stability conditions are

$$M_2 > 0, \quad 1 - \tilde{J}_* > 0, \quad M_1 + 4GI(1 - \tilde{J}_*)/(1 + \sqrt{1 + 16 J_* / J(x)}) > 0$$

The figure shows the stability domain defined by these inequalities in the domain of the parameters $M_1/(GI)$ and $J_*/J(x_1)$ for different values of x_1. The hatching is done in the direction of the stability domain. It can be seen that if the moment of inertia of the engine and the distance x_1 from the point of suspension of the engine to cross-section adjacent to the fuselage are increased, the stability domain of the aerofoil will be reduced.

REFERENCES

1. BAIRAMOV, F. D. and SIRAZETDINOV, T. K., The condition for integral quadratic forms to be sign-definite and for the stability of systems with distributed parameters, *Prikl. Mat. Mekh.*, 1989, **53**, 4, 567–575.
2. BAIRAMOV, F. D., *Stability and Optimal Stabilization of Systems with Distributed Parameters*. Mashinostroyeniye, Moscow, 1995.
3. MIKHAILOV, V. P., *Partial Differential Equations*. Nauka, Moscow, 1983.
4. MOVCHAN, A. A., The stability of processes with respect to two metrics. *Prikl. Mat. Mekh.*, 1960, **24**, 6, 988–1001.
5. SIRAZETDINOV, T. K., *The Stability of Systems with Distributed Parameters*. Nauka, Novosibirsk, 1987

Translated by D.L.